ANSWER KEY

Section A

Q1. (C)

Q2. (B)

Q3. (B)

Q4. (B)

Q5. (C)

Q6. (C)

Q7. (A)

Q8. (C)

Q9. (A)

Q10. (A)

Q11. (C)

Q12. (C)

Q13. (C)

Q14. (C)

Q15. (B)

Q16. (B)

Q17. (C)

Q18. (C)

Q19. (B)

Q20. (B)

Section B

Q21. 009

Q22. 036

Q23. 100

Q24. 396

Q25. 890

DETAILED SOLUTIONS

Section A: Multiple Choice Questions

Q1. Answer: (C)

Let $\alpha = 5+\sqrt{26}$ and $\beta = 5-\sqrt{26}$. Consider the expression $I_n = \alpha^n + \beta^n$. Using the binomial expansion, we have:

$$I_n = (5+\sqrt{26})^n + (5-\sqrt{26})^n = \sum_{k=0}^{n} \binom{n}{k} 5^{n-k}(\sqrt{26})^k + \sum_{k=0}^{n} \binom{n}{k} 5^{n-k}(-\sqrt{26})^k$$

Terms with odd powers of $\sqrt{26}$ cancel out, leaving only integer terms. Thus, I_n is always an integer. Now, observe the value of β:

$$\beta = 5 - \sqrt{26} \approx 5 - 5.099 = -0.099$$

Since $|\beta| < 0.1$, for a large even power $n = 20$, β^{20} is a very small positive number:

$$0 < \beta^{20} < (0.1)^{20} = 10^{-20}$$

We can write $\alpha^{20} = I_{20} - \beta^{20}$. Since I_{20} is an integer and β^{20} is an incredibly small positive number, α^{20} is slightly less than the integer I_{20}. For example, if $I_{20} = K$, then $\alpha^{20} = K - \epsilon$, where ϵ is infinitesimal. This results in a number like $(K-1).99999\ldots$. Consequently, the digits immediately following the decimal point are all 9s. Specifically, the 20th digit is 9.

Q2. Answer: (B)

We are given the functional equation for a polynomial $f(x)$:

$$f(x) \cdot f(1/x) = f(x) + f(1/x)$$

Rearranging this, we get $f(x)f(1/x) - f(x) - f(1/x) + 1 = 1$, which factors to:

$$(f(x) - 1)(f(1/x) - 1) = 1$$

Let $g(x) = f(x) - 1$. Then $g(x)g(1/x) = 1$. Since $f(x)$ is a polynomial, $g(x)$ must be of the form $\pm x^n$. Therefore, $f(x) = 1 \pm x^n$. We are given $f(3) = 28$. Case 1: $1 + 3^n = 28 \implies 3^n = 27 \implies n = 3$. This gives $f(x) = 1 + x^3$. Case 2: $1 - 3^n = 28 \implies -3^n = 27$, which has no integer solution for n. Thus, $f(x) = x^3 + 1$. We need to find $f(4)$:

$$f(4) = 4^3 + 1 = 64 + 1 = 65$$

Q3. Answer: (B)

The equation relating the super ratio x and first term y is:

$$x^2 - 10x + y^2 - 4y + 29 = 0$$

Completing the squares:

$$(x^2 - 10x + 25) + (y^2 - 4y + 4) = 0 \implies (x - 5)^2 + (y - 2)^2 = 0$$

For the sum of squares to be zero, each term must be zero. Hence, $x = 5$ and $y = 2$. A Super GP means $\ln(a_n)$ is a GP. Let the sequence be a_n. $\ln(a_1) = \ln y = \ln 2$. Common ratio of log sequence is $x = 5$. $\ln(a_n) = (\ln 2) \cdot 5^{n-1} = \ln(2^{5^{n-1}})$. So, $a_n = 2^{5^{n-1}}$. We need the remainder of $S_{100} = \sum_{n=1}^{100} 2^{5^{n-1}}$ divided by 17. Powers of 2 mod 17 repeat with period 8 (since $2^4 = 16 \equiv -1, 2^8 \equiv 1$). We must look at the exponent $E_n = 5^{n-1} \pmod 8$. Powers of 5 mod 8: $n = 1 : 5^0 = 1 \pmod 8 \implies 2^1 = 2$. $n = 2 : 5^1 = 5 \pmod 8 \implies 2^5 = 32 \equiv 15 \equiv -2 \pmod{17}$. $n = 3 : 5^2 = 25 \equiv 1 \pmod 8 \implies 2^1 = 2$. $n = 4 : 5^3 \equiv 5 \pmod 8 \implies 2^5 \equiv -2$. The terms alternate between 2 and -2 modulo 17. $S_{100} = (2 - 2) + (2 - 2) + \cdots + (2 - 2)$. There are 50 pairs, and each pair sums to 0. Thus, $S_{100} \equiv 0 \pmod{17}$.

Q4. Answer: (B)

We are given the equation valid for non-zero real x:

$$2f(x) + f(1/x) = 3x \quad \ldots(1)$$

Replace x with $1/x$:

$$2f(1/x) + f(x) = 3(1/x) = \frac{3}{x} \quad \ldots(2)$$

Multiply equation (1) by 2:

$$4f(x) + 2f(1/x) = 6x \quad \ldots(3)$$

Subtract equation (2) from equation (3):

$$(4f(x) + 2f(1/x)) - (f(x) + 2f(1/x)) = 6x - \frac{3}{x}$$

$$3f(x) = 6x - \frac{3}{x} \implies f(x) = 2x - \frac{1}{x}$$

Calculate $f(2)$:

$$f(2) = 2(2) - \frac{1}{2} = 4 - 0.5 = 3.5$$

Q5. Answer: (C)

For regular k-sided polygons to tile the plane, the interior angle must be a divisor of 360° such that an integer number of polygons (n) fit around a vertex. Interior angle of regular k-gon: $\theta = \frac{(k-2)180}{k}$. Condition: $n \cdot \theta = 360$.

$$n \cdot \frac{(k-2)180}{k} = 360 \implies n\frac{k-2}{k} = 2$$

$$n = \frac{2k}{k-2} = \frac{2(k-2)+4}{k-2} = 2 + \frac{4}{k-2}$$

For n to be an integer, $k-2$ must be a divisor of 4. Divisors of 4 are 1, 2, 4. Case 1: $k-2=1 \implies k=3$ (Equilateral Triangle). Case 2: $k-2=2 \implies k=4$ (Square). Case 3: $k-2=4 \implies k=6$ (Hexagon). Sum of possible values of $k = 3+4+6 = 13$.

Q6. Answer: (C)

Let $\angle C = \theta$. Then $\angle B = 2\theta$. In $\triangle ABC$, by the Sine Rule:

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \implies \frac{b}{\sin 2\theta} = \frac{c}{\sin\theta}$$

$$b = \frac{c(2\sin\theta\cos\theta)}{\sin\theta} = 2c\cos\theta$$

Now consider $\triangle ADC$. Let $\angle CAD = \alpha$. Since AD is a bisector, $\angle A = 2\alpha$. The angles in $\triangle ADC$ are $\angle C = \theta$, $\angle CAD = \alpha$, and $\angle ADC = 180^\circ - (\alpha+\theta)$. Using the Sine Rule on $\triangle ADC$:

$$\frac{CD}{\sin\alpha} = \frac{AC}{\sin(180^\circ - (\alpha+\theta))}$$

Given $CD = AB = c$ and $AC = b$, we substitute:

$$\frac{c}{\sin\alpha} = \frac{b}{\sin(\alpha+\theta)}$$

Substitute $b = 2c\cos\theta$:

$$\frac{c}{\sin\alpha} = \frac{2c\cos\theta}{\sin(\alpha+\theta)} \implies \sin(\alpha+\theta) = 2\sin\alpha\cos\theta$$

Expanding the left side:

$$\sin\alpha\cos\theta + \cos\alpha\sin\theta = 2\sin\alpha\cos\theta$$

$$\cos\alpha\sin\theta = \sin\alpha\cos\theta \implies \tan\theta = \tan\alpha$$

Thus, $\alpha = \theta$. Since $\angle A = 2\alpha$, we have $\angle A = 2\theta$. Sum of angles in $\triangle ABC$: $\angle A + \angle B + \angle C = 180^\circ$.

$$2\theta + 2\theta + \theta = 180^\circ \implies 5\theta = 180^\circ \implies \theta = 36^\circ$$

$\angle A = 2\theta = 72^\circ$.

Q7. Answer: (A)

Total outcomes when throwing two dice = $6 \times 6 = 36$. Possible prime sums are 2, 3, 5, 7, 11. Favorable outcomes: Sum = 2: (1,1) [1 way] Sum = 3: (1,2), (2,1) [2 ways] Sum = 5: (1,4), (2,3), (3,2), (4,1) [4 ways] Sum = 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) [6 ways] Sum = 11: (5,6), (6,5) [2 ways] Total favorable ways = $1+2+4+6+2 = 15$. Probability = $\frac{15}{36} = \frac{5}{12}$.

Q8. Answer: (C)

Let $AB = CD = x$ and $BC = DA = y$. Given perimeter condition: $AB + BC + CD = 20 \implies x + y + x = 20 \implies 2x + y = 20 \implies y = 20 - 2x$. E is the midpoint of BC, so $BE = y/2 = 10 - x$. In right-angled $\triangle ABE$ (since ABCD is a rectangle, $\angle B = 90°$):

$$AB^2 + BE^2 = AE^2$$

$$x^2 + (10 - x)^2 = 9^2$$

$$x^2 + 100 - 20x + x^2 = 81$$

$$2x^2 - 20x + 19 = 0$$

We need to find the area of the rectangle: Area $= x \cdot y = x(20 - 2x) = 20x - 2x^2$. From the quadratic equation derived above, we can rearrange to find this value directly:

$$2x^2 - 20x = -19 \implies 20x - 2x^2 = 19$$

Thus, the Area is 19.

Q9. Answer: (A)

To find the number of zeros in base 12 ($12 = 2^2 \cdot 3$), we need to find the exponent of the highest power of 12 dividing $100!$. This is determined by $\min(\lfloor \frac{E_2(100!)}{2} \rfloor, E_3(100!))$, where $E_p(n!)$ is the exponent of prime p in $n!$. Legendre's Formula for 3:

$$E_3(100!) = \lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{9} \rfloor + \lfloor \frac{100}{27} \rfloor + \lfloor \frac{100}{81} \rfloor = 33 + 11 + 3 + 1 = 48$$

Legendre's Formula for 2:

$$E_2(100!) = \lfloor \frac{100}{2} \rfloor + \cdots = 50 + 25 + 12 + 6 + 3 + 1 = 97$$

We need pairs of 2^2, so we have $\lfloor \frac{97}{2} \rfloor = 48$ factors of 2^2. The limiting factor is 48. Thus, there are 48 zeros.

Q10. Answer: (A)

A number is "sticky" if every slicing results in primes. This implies every substring of digits must be prime. Valid digits are $\{2, 3, 5, 7\}$. 1, 4, 6, 8, 9, 0 are not prime. 2 and 5 can only appear at the start (e.g., ...32... gives 32 divisible by 2). Two identical digits are not allowed (e.g., 33 div by 11). We build the set by extending valid primes: Start 2: 2, 23, 237 (comp), 2373... Start 3: 3, 37, 373, 3737 (comp). Start 5: 5, 53, 537 (comp). Start 7: 7, 73, 737 (comp). Let's check primality: 2 (P), 3 (P), 5 (P), 7 (P). 23 (P), 37 (P), 53 (P), 73 (P). 373 (P). Extensions: 23 -> 237 (3×79, X). 37 -> 373. 53 -> 537 (3×179, X). 73 -> 737 (11×67, X). 373 -> 3737 (37×101, X). Set $S = \{2, 3, 5, 7, 23, 37, 53, 73, 373\}$. Cardinality is 9.

Q11. Answer: (C)

Let the side length of the rhombus be s. The rhombus $ADEF$ has vertex A at the top angle of $\triangle ABC$. D is on AB, F is on AC, and E is on BC. $AD = DE = EF = FA = s$. Since $DE \parallel AC$ (property of rhombus), $\triangle DBE \sim \triangle ABC$ is not correct (sides don't align that way). Correct Similarity: Since $EF \parallel AB$, $\triangle EFC \sim \triangle ABC$. Ratio of sides:

$$\frac{EF}{AB} = \frac{FC}{AC}$$

Substitute lengths: $EF = s$, $AB = 15$, $AC = 10$, $FC = AC - AF = 10 - s$.

$$\frac{s}{15} = \frac{10 - s}{10}$$

$$10s = 15(10 - s) \implies 10s = 150 - 15s$$

$$25s = 150 \implies s = 6$$

Q12. Answer: (C)

The set $S = \{1, 5, 9, \ldots, 77\}$. This is an AP with 20 terms. We select 5 numbers and want to maximize |Median − Mean|. Case A: Maximize Mean - Median. We want a small Median. Choose $\{1, 5, 9, 73, 77\}$. Median is 9. Mean $= (1+5+9+73+77)/5 = 165/5 = 33$. Diff $= 33-9 = 24$. Case B: Maximize Median - Mean. We want a large Median. Choose $\{1, 5, 69, 73, 77\}$. Median is 69. Mean $= (1 + 5 + 69 + 73 + 77)/5 = 225/5 = 45$. Diff $= 69 - 45 = 24$. The maximum absolute difference is 24.

Q13. Answer: (C)

Using Mass Point Geometry: D on BC with $BD : DC = 1 : 2$. Assign mass $m_C = 1$, $m_B = 2$. E on AC with $AE : EC = 3 : 2$. To match C, we need mass at C to be a multiple of 2. Let's adjust. Let mass at $C = 6$. Then for BC, $m_B = 3$ (since $3 \cdot 1 \neq 6 \cdot 2$, wait). Balancing BC: $m_B \cdot 1 = m_C \cdot 2$. If $m_C = 3$, $m_B = 6$. Balancing AC: $m_A \cdot 3 = m_C \cdot 2$. With $m_C = 3$, $3m_A = 6 \implies m_A = 2$. Masses: $A = 2, B = 6, C = 3$. P is the center of mass. Line BE: Split AC at E. Mass $E = m_A + m_C = 2 + 3 = 5$. Balancing line BE: $m_B \cdot BP = m_E \cdot PE \implies 6 \cdot BP = 5 \cdot PE$. Ratio $BP : PE = 5 : 6$. Area of $\triangle ABE$: Base $AE = \frac{3}{5}AC$. So Area $= \frac{3}{5} \times 110 = 66$. Area of $\triangle APB$: $\triangle APB$ and $\triangle APE$ share height from A. Bases are on line BE. Area $\triangle APB = \frac{BP}{BE} \times$ Area $\triangle ABE$. $BP/BE = 5/(5 + 6) = 5/11$. Area $= \frac{5}{11} \times 66 = 30$.

Q14. Answer: (C)

Let $\angle A = \theta$. A_1 is the intersection of the bisector of $\angle B$ and bisector of exterior $\angle C$. In $\triangle ABC$, the angle between internal bisector of B and external bisector of C is $\angle A/2$. So $\angle A_1 = \theta/2$. This process repeats for the triangle formed by A_1, B, C. The "new" angle A is A_1. So $\angle A_2 = A_1/2 = \theta/4$. Similarly, $A_k = \theta/2^k$. Sum $= \theta(1/2+1/4+1/8+1/16+1/32) = 93^\circ$. Sum of geometric series $\frac{1}{2}\frac{1-(1/2)^5}{1-1/2} = 1 - 1/32 = 31/32$. $\theta \cdot \frac{31}{32} = 93$. $\theta = \frac{93 \times 32}{31} = 3 \times 32 = 96^\circ$.

Q15. Answer: (B)

Let $|x_i| = 1$ WLOG (since scaling doesn't affect sign). We have k terms equal to $+1$ and $(50 - k)$ terms equal to -1. Let $\Sigma = \sum x_i = k(1) + (50 - k)(-1) = 2k - 50$. We know $(\Sigma)^2 = \sum x_i^2 + 2S = 50 + 2S$. Given $S > 0$ and minimal. Thus $2S = \Sigma^2 - 50 > 0$ and minimal. We need the smallest perfect square $\Sigma^2 > 50$. Candidates: 64, 81, 100... Also $\Sigma = 2k - 50$ is an even number. So Σ^2 must be even. Smallest even square > 50 is 64. $\Sigma^2 = 64 \implies \Sigma = \pm 8$. $2k - 50 = 8 \implies 2k = 58 \implies k = 29$. (Or $2k - 50 = -8 \implies k = 21$). We need to compute $k^2 + (50 - k)^2$: Using $k = 29$: $29^2 + 21^2 = 841 + 441 = 1282$.

Q16. Answer: (B)

$x^2 + y^2 \equiv 0 \pmod 5$. Quadratic residues mod 5 are 0, 1, 4. Possible sums to $0 \pmod 5$: 1) $0 + 0$: $x^2 \equiv 0, y^2 \equiv 0$. $x, y \in \{5, 10, 15, 20\}$. $4 \times 4 = 16$ pairs. 2) $1 + 4$: $x^2 \equiv 1, y^2 \equiv 4$. $x \in \{1, 4, 6, 9, 11, 14, 16, 19\}$ (8 values). $y \in \{2, 3, 7, 8, 12, 13, 17, 18\}$ (8 values). $8 \times 8 = 64$ pairs. 3) $4 + 1$: Symmetric to above. $8 \times 8 = 64$ pairs. Total pairs $= 16 + 64 + 64 = 144$.

Q17. Answer: (C)

$A_1 = \{1, 3, \ldots\}$ (odd numbers, $2k+1$). $A_2 = \{1, 10, \ldots\}$ ($\equiv 1 \pmod 9$). Intersection A_3 starts at 1, common diff $LCM(2, 9) = 18$. $x \in A_3 \implies x \equiv 1 \pmod{18}$. We need perfect squares $m^2 \equiv 1 \pmod{18}$, with $m^2 \leq 2025 \implies m \leq 45$. $m^2 \equiv 1 \pmod{18} \implies m$ is odd and $m^2 \equiv 1 \pmod 9$. $m^2 \equiv 1 \pmod 9 \implies m \equiv 1$ or $8 \pmod 9$. Candidates $m \leq 45$: $1 \pmod 9$: 1, 10, 19, 28, 37. Odd ones: 1, 19, 37. $8 \pmod 9$: 8, 17, 26, 35, 44. Odd ones: 17, 35. Total valid m: $\{1, 17, 19, 35, 37\}$. Total count is 5.

Q18. Answer: (C)

The segment AD is shortest when AD is the altitude to the hypotenuse BC. In a right triangle with altitude AD, $\triangle ABD \sim \triangle CAD$. Ratio of segments $BD/DC = (AB)^2/(AC)^2$. $BD/DC = 10^2/12^2 = 100/144 = 25/36$. Since m, n are natural numbers (assuming coprime for unique ratio representation, or minimal), $m = 25, n = 36$. $m^2 + n^2 = 25^2 + 36^2 = 625 + 1296 = 1921$.

Q19. Answer: (B)

Let edges be a, b, c. $4(a+b+c) = 60 \implies a+b+c = 15$. Surface area $2(ab+bc+ca) = 144$. Define polynomial $P(t) = (t-a)(t-b)(t-c) = t^3 - 15t^2 + 72t - V = 0$, where $V = abc$. For roots to be real, V must lie between the local minimum and local maximum of the cubic $f(t) = t^3 - 15t^2 + 72t$. $f'(t) = 3t^2 - 30t + 72 = 3(t-4)(t-6)$. Critical points at $t = 4$ and $t = 6$. $V_{\text{max}} = f(4) = 4^3 - 15(16) + 72(4) = 64 - 240 + 288 = 112$. $V_{\text{min}} = f(6) = 6^3 - 15(36) + 72(6) = 216 - 540 + 432 = 108$. Difference $V_{\text{max}} - V_{\text{min}} = 112 - 108 = 4$.

Q20. Answer: (B)

Area $= L \times B = 36$. L, B are composite natural numbers. Factor pairs of 36: (1, 36) - 1 is not composite. (2, 18) - 2 is prime. (3, 12) - 3 is prime. (4, 9) - Both composite ($4 = 2^2, 9 = 3^2$). Valid. Perimeter $2(4+9) = 26$. (6, 6) - Both composite. Valid. Perimeter $2(6+6) = 24$. Sum of perimeters $= 26 + 24 = 50$.

Section B: Integer Type Questions

Q21. Answer: 009

The problem asks for the expected number of squares a rook can reach on a 2026×2026 chessboard. A rook moves horizontally or vertically. From any square (r, c), the rook can move to any other square in row r ($2026 - 1$ squares) or any other square in column c ($2026 - 1$ squares). Total reachable squares $N = (2026 - 1) + (2026 - 1) = 4050$. Since N is constant regardless of the rook's position, the expected value is exactly 4050. The question asks for the sum of the digits of the nearest integer. Sum $= 4 + 0 + 5 + 0 = 9$.

Q22. Answer: 036

Using Cauchy-Schwarz Inequality on vectors **u** $= (\sqrt{1/x}, \sqrt{4/y}, \sqrt{9/z})$ and **v** $= (\sqrt{x}, \sqrt{y}, \sqrt{z})$:

$$(\frac{1}{x} + \frac{4}{y} + \frac{9}{z})(x+y+z) \geq (\sqrt{\frac{1}{x}}\sqrt{x} + \sqrt{\frac{4}{y}}\sqrt{y} + \sqrt{\frac{9}{z}}\sqrt{z})^2$$

$$(\text{Expression})(1) \geq (1+2+3)^2 = 6^2 = 36$$

Minimum value is 36.

Q23. Answer: 100

$S_{100} = \sum_{k=1}^{100} k \cdot k!$. Use the identity $k \cdot k! = (k+1)! - k!$. The sum telescopes:

$$S_{100} = (2! - 1!) + (3! - 2!) + \cdots + (101! - 100!) = 101! - 1!$$

$$S_{100} = 101! - 1$$

We need $S_{100} \pmod{101}$. Since 101 is prime (or even just appearing in the factorial product), $101!$ is divisible by 101. $S_{100} \equiv 0 - 1 \equiv -1 \equiv 100 \pmod{101}$.

Q24. Answer: 396

Bed: Swim 4km (1.6 km/h $\rightarrow$ 2.5h), Cycle 180km (30 km/h $\rightarrow$ 6h), Run 42km (7 km/h $\rightarrow$ 6h). Total 14.5h. Rock: Run 42km (10.5 km/h $\rightarrow$ 4h), Cycle 180km (24 km/h $\rightarrow$ 7.5h), Swim 4km (2.5 km/h $\rightarrow$ 1.6h). Total 13.1h. Bed starts cycling at $t = 2.5$. Rock starts cycling at $t = 4$. They meet during the cycling leg. Let x be the distance Rock has cycled when they meet. Bed travels from start of cycle leg, Rock from end (or vice versa, heading towards each other). Time equation: Time(Bed) = Time(Rock). Bed time $= 2.5 + \frac{180-x}{30}$. Rock time $= 4 + \frac{x}{24}$.

$$2.5 + 6 - \frac{x}{30} = 4 + \frac{x}{24}$$

$$8.5 - 4 = x(\frac{1}{30} + \frac{1}{24})$$

$$4.5 = x(\frac{4+5}{120}) = \frac{9x}{120} = \frac{3x}{40}$$

$$x = \frac{4.5 \times 40}{3} = 1.5 \times 40 = 60 \text{ km}$$

Time elapsed when they meet $= 4 + \frac{60}{24} = 4 + 2.5 = 6.5$ hours. Rock's total time to complete race is 13.1 hours. Remaining time for Rock $= 13.1 - 6.5 = 6.6$ hours. Convert to minutes: $6.6 \times 60 = 396$ minutes.

Q25. Answer: 890

We want $n + 10 \mid n^3 + 100$. Use polynomial division logic: $n^3 + 100 = n^2(n + 10) - 10n^2 + 100$ $= n^2(n + 10) - 10n(n + 10) + 100n + 100 = n^2(n + 10) - 10n(n + 10) + 100(n + 10) - 1000 + 100$ $= (n + 10)(n^2 - 10n + 100) - 900$. So, $(n + 10)$ must divide -900. To maximize n, we need to maximize the divisor $n + 10$. The largest divisor of 900 is 900. $n + 10 = 900 \implies n = 890$.